arXiv:1203.3606v2 [math.CO] 27 Dec 2012

GRAPHS OF SMALL RANK-WIDTH ARE PIVOT-MINORS OF GRAPHS OF SMALL TREE-WIDTH

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Abstract. We prove that every graph of rank-width k is a pivot-minor of a graph of tree-width at most $2k$. We also prove that graphs of rank-width at most 1, equivalently distance-hereditary graphs, are exactly vertex-minors of trees, and graphs of linear rank-width at most 1 are precisely vertex-minors of paths. In addition, we show that bipartite graphs of rank-width at most 1 are exactly pivot-minors of trees and bipartite graphs of linear rank-width at most 1 are precisely pivot-minors of paths.

1. Introduction

Rank-width is a width parameter of graphs, introduced by Oum and Seymour [7], measuring how easy it is to decompose a graph into a tree-like structure where the "easiness" is measured in terms of the matrix rank function derived from edges formed by vertex partitions. Rank-width is a generalization of another, more well-known width parameter called tree-width, introduced by Robertson and Seymour [9]. It is well known that every graph of small tree-width also has small rank-width; Oum [8] showed that if a graph has tree-width k, then its rank-width is at most $k+1$. The converse does not hold in general, as complete graphs have rank-width 1 and arbitrary large tree-width.

Pivot-minor and vertex-minor relations are graph containment relations such that rank-width cannot increase when taking pivot-minors or vertex-minors of a graph [7]. Our main result is that for every graph G with rank-width at most k and $|V(G)| \geq 3$, there exists a graph H having G as a pivot-minor such that H has tree-width at most $2k$ and $|V(H)| \leq (2k+1)|V(G)| - 6k$. Furthermore, we prove that for every graph G with linear rank-width at most k and $|V(G)| \geq 3$, there exists a graph H having G as a pivot-minor such that H has path-width at most $k+1$ and $|V(H)| \leq (2k+1)|V(G)| - 6k$.

As a corollary, we give new characterizations of two graph classes: graphs with rank-width at most 1 and graphs with linear rank-width at most 1. We show that a graph has rank-width at most 1 if and only if it is a vertex-minor of a tree. We also prove that a graph has linear rank-width at most 1 if and only if it is a vertex-minor of a path. Moreover, if the graph is bipartite, we prove that a vertex-minor relation can be replaced with a pivot-minor relation in both theorems. Table 1 summarizes our theorems.

Date: January 1, 2013.

Key words and phrases. rank-width, linear rank-width, vertex-minor, pivot-minor, tree-width, path-width, distance-hereditary.

Supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2012-0004119). S. O. is also supported by TJ Park Junior Faculty Fellowship.

G has rank-width $\leq k$	$\Rightarrow$	G is a pivot-minor of a graph of tree-width $\leq 2k$
G has linear rank-width $\leq k$	$\Rightarrow$	G is a pivot-minor of a graph of path-width $\leq k+1$
G has rank-width ≤ 1	$\Leftrightarrow$	G is a vertex-minor of a tree
G has linear rank-width ≤ 1	$\Leftrightarrow$	G is a vertex-minor of a path
G is bipartite and has rank-width ≤ 1	$\Leftrightarrow$	G is a pivot-minor of a tree
G is bipartite and has linear rank-width ≤ 1	$\Leftrightarrow$	G is a pivot-minor of a path

Table 1. Summary of theorems

To prove the main theorem, we construct a graph having G as a pivot-minor, called a rank-expansion. Then we prove that a rank-expansion has small tree-width.

The paper is organized as follows. We present the definition of rank-width and related operations in the next section. In Section 3, we define a *rank-expansion* of a graph and prove the main theorem. In Section 4, using a rank-expansion, we present new characterizations of graphs with rank-width at most 1 and graphs with linear rank-width at most 1.

2. Preliminaries

In this paper, all graphs are simple and undirected. Let $G = (V, E)$ be a graph. For a vertex v of G, let $N(v)$ be the set of vertices adjacent to v and let $\delta(v)$ be the set of edges incident with v. The *degree* of a vertex v, denoted by $\deg(v)$, is defined as $\deg(v) := |\delta(v)|$. For $S \subseteq V$, $G[S]$ denotes the subgraph of G induced on S. For two sets A and B, $A\Delta B = (A \cup B) \setminus (A \cap B)$.

A *vertex partition* of a graph G is a pair (A, B) of subsets of $V(G)$ such that $A \cup B = V(G)$ and $A \cap B = \emptyset$. A vertex $v \in V$ is a *leaf* if $\deg(v) = 1$; Otherwise we call it an *inner vertex*. An edge $e \in E$ is an *inner edge* if e does not have a leaf as an end. Let $V_I(G)$ and $E_I(G)$ be the set of inner vertices of G and inner edges of G, respectively.

For an $X \times Y$ matrix M and subsets $A \subseteq X$ and $B \subseteq Y$, $M[A, B]$ denotes the $A \times B$ submatrix $(m_{i,j})_{i\in A, j\in B}$ of M. For $a \in A$ and $b \in B$, we denote $M_{a,b} = M[\{a\}, \{b\}]$. If $A = B$, then $M[A] = M[A, A]$ is called a *principal submatrix* of M. The adjacency matrix of a graph G, which is a $(0, 1)$-matrix over the binary field, will be denoted by $A(G)$.

Pivoting matrices. Let $M = \begin{array}{c c} & \begin{array}{cc} X & V\setminus X \end{array} \\ \begin{array}{c} X \\ V\setminus X \end{array} & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array}$ be a $V \times V$ matrix over a field F. If $A = M[X]$ is nonsingular, then we define

$$M * X = \begin{array}{c c} & \begin{array}{cc} X & V\setminus X \end{array} \\ \begin{array}{c} X \\ V\setminus X \end{array} & \begin{pmatrix} A^{-1} & A^{-1}B \\ -CA^{-1} & D - CA^{-1}B \end{pmatrix} \end{array}$$

This operation is called a *pivot*, sometimes called a *principal pivot transformation* [10]. Tucker showed the following theorem.

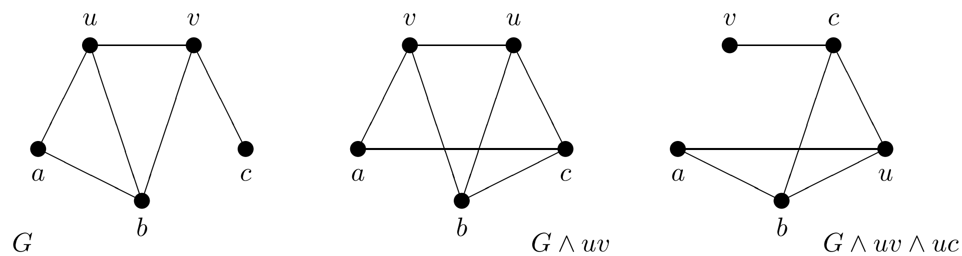

FIGURE 1. Pivoting an edge uv. Note that $G \wedge uv \wedge uc = G \wedge vc$.

Theorem 2.1 (Tucker [11])**.** *Let M be a $V \times V$ matrix over a field. If $M[X]$ is a nonsingular principal submatrix of M, then for every subset Y of V, $(M * X)[Y]$ is nonsingular if and only if $M[X \Delta Y]$ is nonsingular.*

Proof. See Bouchet's proof in Geelen [6, Theorem 2.7]. □

The following thereom is well known, see Geelen [6, Theorem 2.8]. For our purpose, we will only work on skew-symmetric matrices on the binary field and in this case, it follows easily from Theorem 2.1.

Theorem 2.2. *Let M be a square matrix. If $M[X]$ and $M * X[Y]$ are nonsingular, then $(M * X) * Y = M * (X \Delta Y)$.*

Vertex-minors and pivot-minors. The graph obtained from $G = (V, E)$ by applying *local complementation* at a vertex v is

$$G * v = (V, E \Delta \{xy : xv, yv \in E, x \neq y\}).$$

The graph obtained from G by *pivoting* an edge uv is defined by $G \wedge uv = G * u * v * u$.

To see how we obtain the resulting graph by pivoting an edge uv, let $V_1 = N(u) \cap N(v)$, $V_2 = N(u) \setminus (N(v) \cup \{v\})$ and $V_3 = N(v) \setminus (N(u) \cup \{u\})$. One can easily verify that $G \wedge uv$ is identical to the graph obtained from G by complementing adjacency between vertices in distinct sets V_i and V_j and swapping the vertices u and v [7]. See Figure 1 for example.

A graph H is a *vertex-minor* of G if H can be obtained from G by applying a sequence of vertex deletions and local complementations. A graph H is a *pivot-minor* of G if H can be obtained from G by applying a sequence of vertex deletions and pivoting edges. From the definition, every pivot-minor of a graph is a vertex-minor of the graph. Note that every pivot-minor of a bipartite graph is bipartite.

Pivoting in a graph is a special case of a matrix pivot. For a graph G, two vertices u and v are adjacent if and only if $\det(A(G)[\{u, v\}]) \neq 0$. This allows us to determine the graph from the list of nonsingular principal submatrices of $A(G)$. If we are given the list of nonsingular principal submatrices of $A(G) * X$, we can still recover the graph G by Theorem 2.1.

In fact, if $uv \in E$, then $A(G \wedge uv) = A(G) * \{u, v\}$. This is useful, because by Theorem 2.2, the adjacency matrix of $H = G \wedge a_1b_1 \wedge \ldots \wedge a_nb_n$ can be obtained by a single pivot operation $A(G) * X$ where $X = \{a_1, b_1\} \Delta \ldots \Delta \{a_n, b_n\}$. Then u, v are adjacent in H if and only if $A(G)[X \Delta \{u, v\}]$ is nonsingular.

If $A(G)[X]$ is nonsingular, then we denote $G \wedge X$ as the graph having the adjacency matrix $A(G) * X$. For $X \subseteq V(G)$, if $A(G)[X]$ is nonsingular, then we can

obtain the graph $G \wedge X$ from G by applying a sequence of pivoting edges, by Theorem 2.1. Thus, we deduce that H is a pivot-minor of G if and only if $H = G \wedge X \setminus Y$ where $X, Y \subseteq V(G)$ and $A(G)[X]$ is nonsingular.

Rank-width and linear rank-width. The *cut-rank* function $\mathrm{cutrk}_G : 2^V \to \mathbb{Z}$ of a graph $G = (V, E)$ is defined by

$$\mathrm{cutrk}_G(X) = \mathrm{rank}(A(G)[X, V \setminus X]).$$

A tree is *subcubic* if it has at least two vertices and every inner vertex has degree 3. A *rank-decomposition* of a graph G is a pair (T, L), where T is a subcubic tree and L is a bijection from the vertices of G to the leaves of T. For an edge e in T, $T \setminus e$ induces a partition (X_e, Y_e) of the leaves of T. The *width* of an edge e is defined as $\mathrm{cutrk}_G(L^{-1}(X_e))$. The *width* of a rank-decomposition (T, L) is the maximum width over all edges of T. The *rank-width* of G, denoted by $\mathrm{rw}(G)$, is the minimum width of all rank-decompositions of G. If $|V| \leq 1$, then G admits no rank-decomposition and $\mathrm{rw}(G) = 0$.

A tree is a *caterpillar* if it contains a path P such that every vertex of a tree has distance at most 1 to some vertex of P. A *linear rank-decomposition* of a graph G is a rank-decomposition (T, L) of G, where T is a caterpillar. The *linear rank-width* of G is defined as the minimum width of all linear rank-decompositions of G. If $|V| \leq 1$, then G admits no linear rank-decomposition and $\mathrm{lrw}(G) = 0$. Note that if a graph H is a vertex-minor or a pivot-minor of a graph G, then $\mathrm{rw}(H) \leq \mathrm{rw}(G)$ and $\mathrm{lrw}(H) \leq \mathrm{lrw}(G)$ [7]. Trivially, $\mathrm{rw}(G) \leq \mathrm{lrw}(G)$.

Tree-width and path-width. A *tree-decomposition* of a graph $G = (V, E)$ is a pair (T, B) of a tree T and a family $B = \{B_t\}_{t \in V(T)}$ of vertex sets $B_t \subseteq V(G)$, called *bags*, satisfying the following three conditions:

(T1) $V(G) = \bigcup_{v \in V(T)} B_t$.
(T2) For every edge uv of G, there exists a vertex t of T such that u, $v \in B_t$.
(T3) For t_1, t_2 and $t_3 \in V(T)$, $B_{t_1} \cap B_{t_3} \subseteq B_{t_2}$ whenever t_2 is on the path from t_1 to t_3.

The *width* of a tree-decomposition (T, B) is $\max\{|B_t| - 1 : t \in V(T)\}$. The *tree-width* of G, denoted by $\mathrm{tw}(G)$, is the minimum width of all tree-decompositions of G. A *path-decomposition* of a graph G is a tree-decomposition (T, B) where T is a path. The *path-width* of G, denoted by $\mathrm{pw}(G)$, is the minimum width of all path-decompositions of G.

3. Rank-expansions and pivot-minors of graphs with small tree-width

In this section, we aim to construct, for a graph G of rank-width k, a bigger graph having tree-width at most $2k$ such that it has a pivot-minor isomorphic to G.

Theorem 3.1. *Let k be a non-negative integer. Let G be a graph of rank-width at most k such that $|V(G)| \geq 3$. Then there exists a graph H having a pivot-minor isomorphic to G such that tree-width of H is at most $2k$ and $|V(H)| \leq (2k+1)|V(G)| - 6k$.*

For a graph of small linear rank-width, we can find a bigger graph having small path-width instead of tree-width and reduce the upper bound on the path-width of a bigger graph as follows.

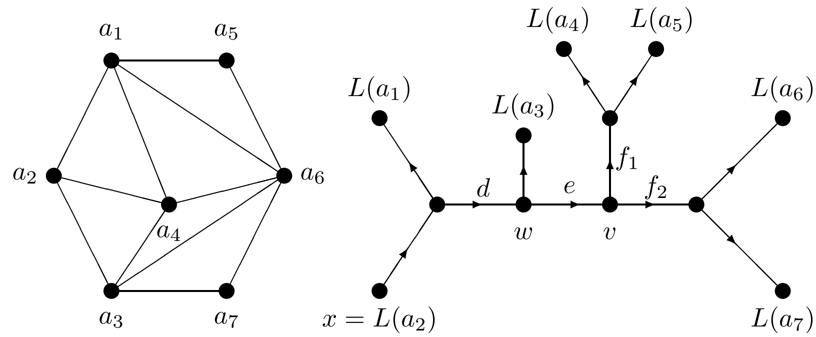

FIGURE 2. A graph G and a rank-decomposition (T, L) of G with a fixed leaf $x \in V(T)$. Note that the edge $e \in E(T)$ has width 3 and e is directed from w to v.

Theorem 3.2. *Let k be a non-negative integer. Let G be a graph of linear rank-width at most k and $|V(G)| \geq 3$. Then there exists a graph H having a pivot-minor isomorphic to G such that path-width of H is at most $k+1$ and $|V(H)| \leq (2k+1)|V(G)| - 6k$.*

To prove these two theorems, we need the following simple lemma on linear algebra.

Lemma 3.3. *Let G be a graph and (A_1, B_1), (A_2, B_2) be two vertex partitions of G such that $A_2 \subseteq A_1$. Let $S \subseteq A_1$ be a set corresponding to a basis of row vectors in $A(G)[A_1, B_1]$. Then there exists a subset of A_2 representing a basis of row vectors in $A(G)[A_2, B_2]$ containing $S \cap A_2$.*

Proof. Because $A_2 \subseteq A_1$, row vectors in $A(G)[S \cap A_2, B_2]$ are linearly independent. Therefore we can extend $S \cap A_2$ to a basis of rows in $A(G)[A_2, B_2]$. □

3.1. **Construction of a rank-expansion.** To prove Theorems 3.1 and 3.2, we construct a *rank-expansion* of a graph as follows. Let G be a connected graph and (T, L) be a rank-decomposition of G having width at most k. We fix a leaf $x \in V(T)$. For $e \in E(T)$, let T_e be the component of $T \setminus e$ which does not contain x, and let $A_e = L^{-1}(V(T_e))$, $B_e = V(G) \setminus A_e$ and $M_e = A(G)[A_e, B_e]$. For each $a \in A_e$, let $R^e_a = M_e[\{a\}, B_e]$ be the row vector of M_e corresponding to a.

First, we orient each edge of T away from x. By Lemma 3.3, we can choose a vertex set $U_e \subseteq A_e$ for each edge e of T satisfying the following two conditions:

(1) $\{R^e_w\}_{w \in U_e}$ forms a basis of row vectors in M_e for each edge e of T.
(2) $(U_e \cap A_f) \subseteq U_f$ if the tail of an edge f is the head of e.

Since (T, L) has width at most k, we have $|U_e| \leq k$ for each edge e of T. Since R^e_a can be uniquely expressed as a linear combination of vectors in $\{R^e_w\}_{w \in U_e}$ for each $a \in A_e$, there exists a unique $A_e \times U_e$ matrix P_e such that $P_e(A(G)[U_e, B_e]) = A(G)[A_e, B_e]$.

For example, in Figure 2,

$$A(G)[A_e, B_e] = \begin{array}{c c} & \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{array}{c} a_4 \\ a_5 \\ a_6 \\ a_7 \end{array} & \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

and $\{R^e_{a_4}, R^e_{a_5}, R^e_{a_7}\}$ forms a basis of row vectors of $A(G)[A_e, B_e]$. So, if we let $U_e = \{a_4, a_5, a_7\}$, then

$$P_e = \begin{array}{c c} & \begin{array}{ccc} a_4 & a_5 & a_7 \end{array} \\ \begin{array}{c} a_4 \\ a_5 \\ a_6 \\ a_7 \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

and we easily verify that $P_e A(G)[U_e, B_e] = A(G)[A_e, B_e]$.

If the tail of an edge f is the head of an edge e, then let $C_f = P_e[U_f, U_e]$. We will use the property that if $e_{n+1}e_n \dots e_1$ is a directed path in T, then

$$C_{e_1} C_{e_2} \dots C_{e_n} = P_{e_{n+1}}[U_{e_1}, U_{e_{n+1}}].$$

A *rank-expansion* $\boldsymbol{R}(G, T, L, x, \{U_f\}_{f \in E(T)})$ of a graph G is a graph H such that

$$V(H) = \bigcup_{v \in V_I(T)} S_v \quad \text{ where } S_v = \bigcup_{e \in \delta(v)} (U_e \times \{e\} \times \{v\}) \text{ for each } v \in V_I(T),$$

$$\begin{aligned} E(H) = \{\{(a,e,v),(a,e,w)\} &: e = vw \in E_I(T), a \in U_e\} \\ \cup \{\{(a,e,v),(b,f,v)\} &: v \in V_I(T), e, f \in E(T), v \text{ is the head of } e \text{ and the tail of } f, \\ &\qquad a \in U_f, b \in U_e \text{ and } (C_f)_{a,b} \neq 0\} \\ \cup \{\{(a,f_1,v),(b,f_2,v)\} &: v \text{ is the tail of both } f_1 \text{ and } f_2 \in E(T), \\ &\qquad a \in U_{f_1}, b \in U_{f_2} \text{ and } ab \in E(G)\}. \end{aligned}$$

(The sets $V_I(T), E_I(T)$ are defined in the beginning of Section 2.)

For $e = vw \in E_I(T)$, let $\overline{e} = \{(a,e,v) : a \in U_e\} \cup \{(a,e,w) : a \in U_e\} \subseteq V(H)$ and for $W \subseteq E_I(T)$, let $\overline{W} = \bigcup_{e \in W} \overline{e} \subseteq V(H)$. If $e \in E_I(T)$ is directed from w to v, let $L_e = S_v \cap \overline{e}$ and $R_e = S_w \cap \overline{e}$. For a vertex a in $V(G)$, T has a unique edge e incident with $L(a)$ and some vertex v of T and we write $\overline{a}$ to denote the unique vertex in $U_e \times \{e\} \times \{v\}$ and let $\overline{e} := \overline{a}$. Notice that since G is connected, U_e is nonempty.

We discuss the number of vertices in the rank-expansion H. We easily observe that $|E_I(T)| = |V(G)| - 3$. So if $\operatorname{rw}(G) \le k$, then $|\overline{e}| \le 2k$ for each $e \in E_I(T)$, and we deduce that $|V(H)| \le 2k|E_I(T)| + |V(G)| = 2k(|V(G)| - 3) + |V(G)| = (2k+1)|V(G)| - 6k$.

3.2. **A graph is a pivot-minor of its rank-expansion.** First, we prove that every rank-expansion of a graph G has a pivot-minor isomorphic to G. To obtain G as a pivot-minor of a rank-expansion H, we will prove that $H \wedge \overline{E_I(T)}$ has an induced subgraph isomorphic to G. We first need to verify that $A(H)[\overline{E_I(T)}]$ is nonsingular in order to apply the matrix pivot.

Lemma 3.4. *Let G be a graph and $uv \in E(G)$. If $\deg(u) = 1$, then $G \wedge uv \setminus \{u, v\} = G \setminus \{u, v\}$.*

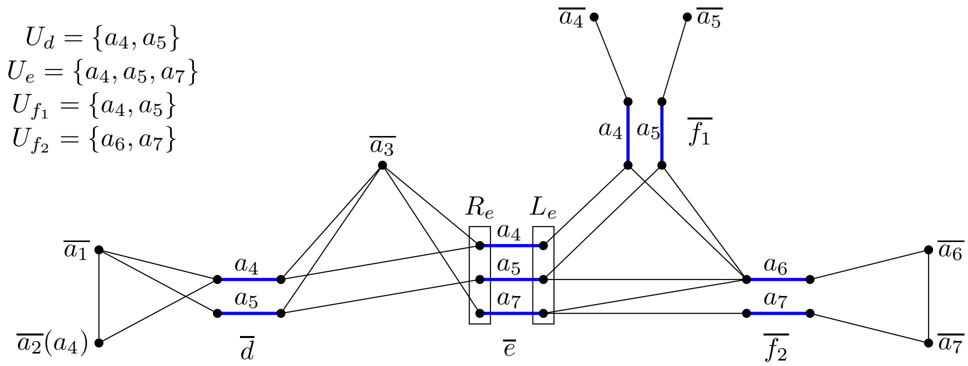

FIGURE 3. A rank-expansion of the graph G in Figure 2.

Proof. It is clear from the definition. □

Lemma 3.5. *The matrix $A(H)[\overline{E_I(T)}]$ is nonsingular.*

Proof. We claim that for all $W \subseteq E_I(T)$, $A(H)[\overline{W}]$ is nonsingular. We proceed by induction on $|W|$. If W is empty, then it is trivial. If $|W| \geq 1$, then W induces a forest in T, and therefore there must be an edge $f \in W$ which has a leaf in $T[W]$. By induction hypothesis, $A(H)[\overline{W \setminus \{f\}}]$ is nonsingular. Since every edge in $H[\overline{f}]$ is incident with a leaf in $H[\overline{W}]$, by Lemma 3.4, pivoting all edges in $\overline{f}$ does not change the graph $H[\overline{W \setminus \{f\}}]$. So, $A(H[\overline{W}] \wedge \overline{f})[\overline{W \setminus \{f\}}] = A(H)[\overline{W \setminus \{f\}}]$ and therefore, by Theorem 2.1, $A(H)[\overline{f} \Delta \overline{W \setminus \{f\}}] = A(H)[\overline{W}]$ is nonsingular. □

By Lemma 3.5, we can pivot H by $\overline{E_I(T)}$. Now in order to determine the adjacency in the graph $H \wedge \overline{E_I(T)}$, we need to determine whether the matrix $A(H)[\overline{E_I(T)} \cup \{\overline{a}, \overline{b}\}]$ is nonsingular where a, $b \in V(G)$. In the following lemma, we will show that to determine the adjacency in the graph $H \wedge \overline{E_I(T)}$, it is enough to pivot a small set of vertices.

Lemma 3.6. *Let a, $b \in V(G)$ and let P be a path from $L(a)$ to $L(b)$ in T. Then $A(H)[\overline{E_I(T)} \cup \{\overline{a}, \overline{b}\}]$ is nonsingular if and only if $A(H)[\overline{E(P)}]$ is nonsingular.*

Proof. We claim that for $E(P) \cap E_I(T) \subseteq W \subseteq E_I(T)$, $A(H)[\overline{W} \cup \{\overline{a}, \overline{b}\}]$ is nonsingular if and only if $A(H)[\overline{E(P)}]$ is nonsingular.

We use induction on $|W|$. If $W = E(P) \cap E_I(T)$, then it is trivial, because $\overline{W} \cup \{\overline{a}, \overline{b}\} = \overline{E(P)}$. So we may assume that $|W| > |E(P) \cap E_I(T)|$. Since P is a maximal path in T, the subgraph of T having the edge set $W \cup E(P)$ must have at least 3 leaves. Thus there is an edge f in $W \setminus E(P)$ incident with a leaf in $T[W \cup E(P)]$ other than $L(a)$ and $L(b)$. Since every edge in $\overline{f}$ is incident with a leaf in $H[\overline{W}]$, by Lemma 3.4, $A(H[\overline{W} \cup \{\overline{a}, \overline{b}\}] \wedge \overline{f})[\overline{W \setminus \{f\}} \cup \{\overline{a}, \overline{b}\}] = A(H)[\overline{W \setminus \{f\}} \cup \{\overline{a}, \overline{b}\}]$. By induction hypothesis and Theorem 2.1, we deduce that

$$\begin{aligned} A(H)[\overline{E(P)}] \text{ is nonsingular} &\Leftrightarrow A(H)[\overline{W \setminus \{f\}} \cup \{\overline{a}, \overline{b}\}] \text{ is nonsingular} \\ &\Leftrightarrow A(H[\overline{W} \cup \{\overline{a}, \overline{b}\}] \wedge \overline{f})[\overline{W \setminus \{f\}} \cup \{\overline{a}, \overline{b}\}] \text{ is nonsingular} \\ &\Leftrightarrow A(H)[\overline{W} \cup \{\overline{a}, \overline{b}\}] \text{ is nonsingular.} \end{aligned}$$ □

From now on, we focus on how to determine the adjacency in $H \wedge \overline{E_I(T)}$ by computing $\det\Big(A(H)[\overline{E(P)}]\Big)$.

Lemma 3.7. *Let $P = (e_{n+1}, e_n, \ldots, e_1)$ be the directed path from w to v in T. Then $C_{e_1} C_{e_2} \ldots C_{e_n} A(G)[U_{e_{n+1}}, B_{e_{n+1}}] = A(G)[U_{e_1}, B_{e_{n+1}}]$.*

Proof. We proceed by induction on n. If $n = 1$, then by definition,

$$C_{e_1} A(G)[U_{e_2}, B_{e_2}] = P_{e_2}[U_{e_1}, U_{e_2}] A(G)[U_{e_2}, B_{e_2}] = A(G)[U_{e_1}, B_{e_2}].$$

We may assume that $n \geq 2$. By induction hypothesis,

$$C_{e_2} C_{e_3} \ldots C_{e_n} A(G)[U_{e_{n+1}}, B_{e_{n+1}}] = A(G)[U_{e_2}, B_{e_{n+1}}].$$

Since $C_{e_1} A(G)[U_{e_2}, B_{e_2}] = A(G)[U_{e_1}, B_{e_2}]$ and $B_{e_{n+1}} \subseteq B_{e_2}$,

$$C_{e_1} A(G)[U_{e_2}, B_{e_{n+1}}] = A(G)[U_{e_1}, B_{e_{n+1}}].$$

Therefore, we conclude that

$$\begin{aligned} C_{e_1} C_{e_2} \ldots C_{e_n} A(G)[U_{e_{n+1}}, B_{e_{n+1}}] &= C_{e_1} A(G)[U_{e_2}, B_{e_{n+1}}] \\ &= A(G)[U_{e_1}, B_{e_{n+1}}]. \end{aligned}$$ □

Lemma 3.8.

$$\det \left(\begin{array}{c|cccccc} 0 & C_1 & 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & I & C_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & C_3 & & 0 & 0 \\ 0 & 0 & 0 & I & & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I & C_n \\ C_{n+1} & 0 & 0 & 0 & \cdots & 0 & I \end{array} \right) = (-1)^n \det(C_1 C_2 \ldots C_{n+1}).$$

(*Since we mainly focus on the binary field,* $-1 = +1$.)

Proof. By elementary row operation,

$$\det \left(\begin{array}{c|cccccc} 0 & C_1 & 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & I & C_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & C_3 & & 0 & 0 \\ 0 & 0 & 0 & I & & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I & C_n \\ C_{n+1} & 0 & 0 & 0 & \cdots & 0 & I \end{array} \right)$$

$$= \det \left(\begin{array}{c|cccccc} 0 & 0 & -C_1 C_2 & 0 & \cdots & 0 & 0 \\ \hline 0 & I & C_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & C_3 & & 0 & 0 \\ 0 & 0 & 0 & I & & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I & C_n \\ C_{n+1} & 0 & 0 & 0 & \cdots & 0 & I \end{array} \right)$$

$$= \det \left(\begin{array}{c|cccccc} 0 & 0 & 0 & (-1)^2 C_1C_2C_3 & \cdots & 0 & 0 \\ \hline 0 & I & C_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & C_3 & & 0 & 0 \\ 0 & 0 & 0 & I & & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I & C_n \\ C_{n+1} & 0 & 0 & 0 & \cdots & 0 & I \end{array}\right)$$

$$= \det \left(\begin{array}{c|cccccc} (-1)^n C_1C_2\dots C_{n+1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & I & C_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & C_3 & & 0 & 0 \\ 0 & 0 & 0 & I & & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I & C_n \\ C_{n+1} & 0 & 0 & 0 & \cdots & 0 & I \end{array}\right)$$

$$= (-1)^n \det(C_1C_2\dots C_{n+1}). \qquad \square$$

Proposition 3.9. *Let $k \geq 1$. Let G be a connected graph with rank-width k and $|V(G)| \geq 3$. Then a rank-expansion of G has a pivot-minor isomorphic to G.*

Proof. Let (T, L) be a rank-decomposition of a graph G and let x be a leaf in T. We orient each edge f away from x. For each $f \in E(T)$, if m is the width of f, we choose a basis $U_f = \{u_1^f, u_2^f, \dots, u_m^f\} \subseteq A_f$ of rows in the matrix $A(G)[A_f, B_f]$ such that $(U_e \cap A_f) \subseteq U_f$ if the head of an edge e is the tail of f. Since G is connected, $|U_f| \geq 1$. Let H be a rank-expansion $\boldsymbol{R}(G, T, L, x, \{U_f\}_{f\in E(T)})$ of a graph G. By Lemma 3.5, $A(H)[\overline{E_I(T)}]$ is nonsingular. We will prove that for a, $b \in V(G)$, $\overline{a}\overline{b} \in E(H \wedge \overline{E_I(T)})$ if and only if $ab \in E(G)$.

Let a, b be distinct vertices in G. We consider the path P from $L(a)$ to $L(b)$ in T. By Theorem 2.1 and Lemma 3.6,

$$\begin{aligned}\left(A(H \wedge \overline{E_I(T)})\right)_{\overline{a},\overline{b}} &= \det\left(A(H \wedge \overline{E_I(T)})[\{\overline{a}, \overline{b}\}]\right) \\ &= \det\left(A(H)[\overline{E_I(T)} \Delta \{\overline{a}, \overline{b}\}]\right) = \det\left(A(H)[\overline{E(P)}]\right).\end{aligned}$$

Thus, it is enough to show that $\det(A(H[\overline{E(P)}])) = (A(G))_{a,b}$.

If $L(b) = x$, then $P = (e_{n+1}, e_n, \dots, e_1, e_0)$ is a directed path from $L(b)$ to $L(a)$. The submatrix of $A(H)$ induced by $\overline{E(P)}$ is

$$
\begin{array}{cc}
 & \begin{array}{c|ccccc|c|ccccc} \overline{b} & L_{e_1} & L_{e_2} & \cdots & L_{e_{n-1}} & L_{e_n} & \overline{a} & R_{e_1} & R_{e_2} & \cdots & R_{e_{n-1}} & R_{e_n} \end{array} \\
\begin{array}{c} \overline{a} \\ R_{e_1} \\ R_{e_2} \\ \vdots \\ R_{e_{n-1}} \\ R_{e_n} \\ \overline{b} \\ L_{e_1} \\ L_{e_2} \\ \vdots \\ L_{e_{n-1}} \\ L_{e_n} \end{array} &
\left(\begin{array}{c|ccccc|c|ccccc}
0 & C_{e_0} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \hline
0 & I & C_{e_1} & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & I & & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & & & \ddots & & \vdots & 0 & & & \ddots & & \vdots \\
0 & 0 & 0 & \cdots & I & C_{e_{n-1}} & 0 & 0 & 0 & \cdots & 0 & 0 \\
C_{e_n} & 0 & 0 & \cdots & 0 & I & 0 & 0 & 0 & \cdots & 0 & 0 \\ \hline
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & C_{e_n}^t \\ \hline
0 & 0 & 0 & \cdots & 0 & 0 & C_{e_0}^t & I & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & C_{e_1}^t & I & & 0 & 0 \\
\vdots & & & \ddots & & \vdots & 0 & & & \ddots & & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & I & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & C_{e_{n-1}}^t & I
\end{array}\right)
\end{array}
$$

$$
= \left(\begin{array}{c|c} C & 0 \\ \hline 0 & C^t \end{array}\right).
$$

Note that $\det(A(H)[\overline{E(P)}]) = \det(C)\det(C^t) = \det(C)^2$. By Lemma 3.8,

$$\det(C) = (-1)^n \det(C_{e_0} C_{e_1} \dots C_{e_n}).$$

Since $|U_{e_{n+1}}| = |B_{e_{n+1}}| = 1$ and $\operatorname{rank}(A(G)[U_e, B_e]) = |U_e|$ for all edges $e \in E(T)$, $A(G)[U_{e_{n+1}}, B_{e_{n+1}}] = (1)$. By Lemma 3.7,

$$
\begin{aligned}
C_{e_0} C_{e_1} \dots C_{e_n} &= C_{e_0} C_{e_1} \dots C_{e_n} A(G)[U_{e_{n+1}}, B_{e_{n+1}}] \\
&= A(G)[U_{e_0}, B_{e_{n+1}}] \\
&= (A(G))_{a,b}.
\end{aligned}
$$

Therefore $\det(A(H)[\overline{E(P)}]) = (A(G))_{a,b}$, as required.

Now we assume that $L(a) \neq x$ and $L(b) \neq x$. Then there exists a vertex y in $V(P)$ such that it has a shortest distance to x. Let $P_1 = (e_n, e_{n-1}, \dots, e_0)$ be the edges of P from y to $L(a)$ and $P_2 = (f_m, f_{m-1}, \dots, f_0)$ be the edges of P from y to $L(b)$.

Let $M = A(H)[R_{e_n}, R_{f_m}]$. By the construction of a rank-expansion, $M = A(G)[U_{e_n}, U_{f_m}]$. The submatrix of $A(H)$ induced by $\overline{E(P)}$ is

$$
\begin{array}{cc}
 & \begin{array}{c|c} \{\overline{b}\} \cup \bigcup_{i=1}^n L_{e_i} \cup \bigcup_{i=1}^m R_{f_i} & \{\overline{a}\} \cup \bigcup_{i=1}^n R_{e_i} \cup \bigcup_{i=1}^m L_{f_i} \end{array} \\
\begin{array}{c} \{\overline{a}\} \cup \bigcup_{i=1}^n R_{e_i} \cup \bigcup_{i=1}^m L_{f_i} \\ \{\overline{b}\} \cup \bigcup_{i=1}^n L_{e_i} \cup \bigcup_{i=1}^m R_{f_i} \end{array} &
\left(\begin{array}{c|c} C & 0 \\ \hline 0 & C^t \end{array}\right)
\end{array}
$$

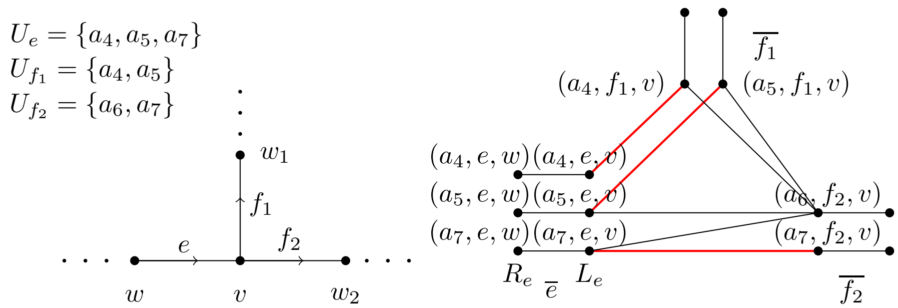

FIGURE 4. A rank-expansion of the graph G in Figure 2. By the construction of a rank-expansion, every vertex in L_e has exactly one neighbor in $R_{f_1} \cup R_{f_2} \setminus \{(a_6, f_2, v)\}$ in the subgraph $H[S_v]$.

where C is

$$
\begin{array}{cc}
 & \begin{array}{c|ccccc|ccccc} \overline{b} & L_{e_1} & L_{e_2} & \cdots & L_{e_{n-1}} & L_{e_n} & R_{f_m} & R_{f_{m-1}} & \cdots & R_{f_2} & R_{f_1} \end{array} \\
\begin{array}{c} \overline{a} \\ R_{e_1} \\ R_{e_2} \\ \vdots \\ R_{e_{n-1}} \\ R_{e_n} \\ L_{f_m} \\ L_{f_{m-1}} \\ \vdots \\ L_{f_2} \\ L_{f_1} \end{array} &
\left(\begin{array}{c|ccccc|ccccc}
0 & C_{e_0} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \hline
0 & I & C_{e_1} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & I & & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & & & \ddots & & \vdots & & & \ddots & & \vdots \\
0 & 0 & 0 & \cdots & I & C_{e_{n-1}} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & I & M & 0 & \cdots & 0 & 0 \\ \hline
0 & 0 & 0 & \cdots & 0 & 0 & I & C^t_{f_{m-1}} & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & I & & 0 & 0 \\
\vdots & & & \ddots & & \vdots & & & \ddots & & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & I & C^t_{f_1} \\
C^t_{f_0} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & I
\end{array}\right).
\end{array}
$$

It is enough to show that $C_{e_0}C_{e_1}\ldots C_{e_{n-1}}MC^t_{f_{m-1}}C^t_{f_{m-2}}\ldots C^t_{f_0} = A(G)(a,b)$. Since $M = A(G)[U_{e_n}, U_{f_m}] \subseteq A(G)[U_{e_n}, B_{e_n}]$, by Lemma 3.7, we have

$$
\begin{aligned}
&C_{e_0}C_{e_1}\ldots C_{e_{n-1}}MC^t_{f_{m-1}}C^t_{f_{m-2}}\ldots C^t_{f_0} \\
&= C_{e_0}C_{e_1}\ldots C_{e_{n-1}}A(G)[U_{e_n}, U_{f_m}]C^t_{f_{m-1}}C^t_{f_{m-2}}\ldots C^t_{f_0} \\
&= A(G)[U_{e_0}, U_{f_m}]C^t_{f_{m-1}}C^t_{f_{m-2}}\ldots C^t_{f_0} \\
&= (C_{f_0}C_{f_1}\ldots C_{f_{m-1}}A(G)[U_{f_m}, U_{e_0}])^t \\
&= A(G)[U_{f_0}, U_{e_0}]^t = (A(G))_{a,b}.
\end{aligned}
$$

So, $\det(A(H)[\overline{E(P)}]) = (A(G))_{a,b}$, as claimed. Therefore, $\overline{a}\overline{b} \in E(H \wedge \overline{E_I(T)})$ if and only if $ab \in E(G)$. We conclude that a rank-expansion of G has a pivot-minor isomorphic to G. $\square$

3.3. **A rank-expansion has small tree-width.** In the next proposition, we show that a rank-expansion has tree-width at most $2k$ when $\mathrm{rw}(G) \leq k$.

Proposition 3.10. *Let $k \geq 1$. Let G be a connected graph with $|V(G)| \geq 3$. If G has rank-width k, Then G has a rank-expansion of tree-width at most $2k$. Moreover,*

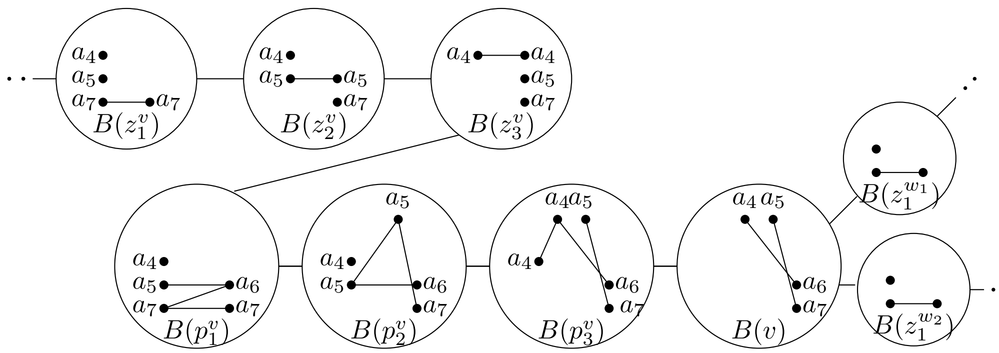

FIGURE 5. Tree-decomposition of a rank-expansion in Figure 4. The vertex sets $B(z_i^v)$ and $B(p_i^v)$, defined in Proposition 3.9, are bags which decompose $H[\overline{e}]$ and $H[S_v]$, respectively.

if G has linear rank-width k, then G has a rank-expansion of path-width at most $k+1$.

Proof. Let (T, L) be a rank-decomposition of G of width k. We fix a leaf $x \in V(T)$ and orient each edge f away from x. For each $f \in E(T)$, if m is the width of f, we choose a basis $U_f = \{u_1^f, u_2^f, \ldots, u_m^f\} \subseteq A_f$ of rows in the matrix $A(G)[A_f, B_f]$ such that $(U_e \cap A_f) \subseteq U_f$ if the head of an edge e is the tail of f. Since G is connected, $|U_f| \geq 1$. Let H be a rank-expansion $\boldsymbol{R}(G, T, L, x, \{U_f\}_{f \in E(T)})$ of a graph G.

Let T' be a tree obtained from $T[V_I(T)]$ by replacing each edge from w to v with a path $w z_1^v z_2^v \ldots z_{|U_e|}^v p_1^v p_2^v \ldots p_{|U_e|}^v v$. Let y be the neighbor of x in T and let $B(y) = S_y$. For $v \in V_I(T) \setminus \{y\}$, let $e = vw$ be the edge incoming to v and f_1, f_2 be edges outgoing from v. Let $R^v = \{(a, f, v) \in R_{f_1} \cup R_{f_2} : a \notin U_e\}$. Since $(U_e \cap A_{f_i}) \subseteq U_{f_i}$ for each $i \in \{1, 2\}$, each vertex in L_e has exactly one neighbor in $R_{f_1} \cup R_{f_2} \setminus R^v$. Let $B(v) = R_{f_1} \cup R_{f_2}$ and $B(z_1^v) = R_e \cup \{(u_1^e, e, v)\}$, $B(p_1^v) = R^v \cup L_e \cup \{(a, f, v) \in R_{f_1} \cup R_{f_2} : a = u_1^e\}$. And for each $2 \leq i \leq |U_e|$, we define

$$B(z_i^v) = B(z_{i-1}^v) \setminus \{(u_{i-1}^e, e, w)\} \cup \{(u_i^e, e, v)\}$$
$$B(p_i^v) = B(p_{i-1}^v) \setminus \{(u_{i-1}^e, e, v)\} \cup \{(a, f, v) \in R_{f_1} \cup R_{f_2} : a = u_i^e\}.$$

Now we show that the pair $(T', \{B(v)\}_{v \in V(T')})$ is a tree-decomposition of H. Note that for each $v \in V_I(T) \setminus \{y\}$ with the incoming edge e, $\bigcup_i E(H[B(z_i^v)]) = E(H[\overline{e}])$ and $\bigcup_i E(H[B(p_i^v)]) = E(H[S_v])$. Therefore all vertices and all edges in H are covered by $B(v)$ for some $v \in V(T')$. So the first and second axioms of a tree-decomposition are satisfied.

For the third axiom, it suffices to show that for every $t \in V(H)$, $T'[\{z : B(z) \ni t\}]$ is a subtree of T'. Let $t = (u_j^e, e, v) \in V(H)$ for some $e = vw \in E(T)$ and $1 \leq j \leq |U_e|$. If v is the head of e, $T'[\{z : B(z) \ni t\}] = T'[\{z_j^v, \ldots, z_{|U_e|}^v, p_1^v, \ldots, p_j^v\}]$, and it forms a path. Suppose v is the tail of e. Let f be the edge incoming to v, and if $a \in U_f$, then let h be the integer such that $a = u_h^f$, if otherwise, let $h = 1$. Then $T'[\{z : B(z) \ni t\}] = T'[\{p_h^v, \ldots, p_{|U_e|}^v, v, z_1^w, \ldots, z_j^w\}]$. It also forms a path, thus $(T', \{B(v)\}_{v \in V(T')})$ is a tree-decomposition of H.

Since $|B(y)| \leq 2k+1$ and for each $v \in V_I(T) \setminus \{y\}$ with the incoming edge e, $|B(z_i^v)| = |B(z_1^v)| = |R_e| + 1 \leq k+1$, $|B(p_i^v)| = |B(p_1^v)| = |R^v| + |L_e| + 1 \leq (2k - |U_e|) + |U_e| + 1 = 2k+1$ and $|B(v)| \leq 2k$, the resulting tree-decomposition has width at most $2k$.

Suppose that G has linear rank-width at most k. Here, we choose $x \in V(T)$ such that x is an end of a longest path in T, and let y be the neighbor of x. For $v \in V_I(T)$ with outgoing edges f_1 and f_2, $|U_{f_1}| = 1$ or $|U_{f_2}| = 1$ because every inner vertex of T is incident with a leaf. Therefore, for each $v \in V_I(T) \setminus \{y\}$ and $1 \leq i \leq |U_e|$, $|B(p_i^v)| \leq (k+1-|U_e|) + |U_e| + 1 = k+2$ and $|B(v)| \leq k+1$, and $|B(y)| \leq k+2$. Moreover, since $T[V_I(T)]$ is a path, T' is also a path. Therefore $(T', \{B(v)\}_{v \in V(T')})$ is a path-decomposition of H with path-width at most $k+1$. □

Proof of Theorem 3.1. If $k = 0$, then it is trivial. We assume that $k \geq 1$. We proceed by induction on the number of vertices.

Suppose G is connected. Since G has rank-width at most k and $|V(G)| \geq 3$, by Proposition 3.10, there is a rank-expansion H of G such that $\mathrm{tw}(H) \leq 2k$, and $|V(H)| \leq (2k+1)|V(G)| - 6k$. By Proposition 3.9, H has a pivot-minor isomorphic to G.

If G is disconnected, then we choose a largest component Y of G. Since $k \geq 1$, the component Y has at least 2 vertices. If $|V(Y)| = 2$, then G has rank-width 1 and tree-width 1, and $|V(G)| \leq (2+1)|V(G)| - 6$ since $|V(G)| \geq 3$. We assume that $|V(Y)| \geq 3$. Then by induction hypothesis, there is a graph H_1 such that Y is isomorphic to a pivot-minor of H_1 and $\mathrm{tw}(H_1) \leq 2k$ and $|V(H_1)| \leq (2k+1)|V(Y)| - 6k$.

If $G \setminus V(Y)$ has tree-width at most 1, then G is isomorphic to a pivot-minor of the disjoint union of two graphs H_1 and $G \setminus V(Y)$, and the tree-width of it is equal to the tree-width of H_1. Since $|V(H_1)| + |V(G) \setminus V(Y)| \leq (2k+1)|V(Y)| - 6k + |V(G) \setminus V(Y)| \leq (2k+1)|V(G)| - 6k$, we obtain the result. If tree-width of $G \setminus V(Y)$ is at least 2, then $|V(G) \setminus V(Y)| \geq 3$. Therefore, by induction hypothesis, there is a graph H_2 such that $G \setminus V(Y)$ is isomorphic to a pivot-minor of H_2 and $\mathrm{tw}(H_2) \leq 2k$ and $|V(H_2)| \leq (2k+1)|V(G) \setminus V(Y)| - 6k$. So G is isomorphic to a pivot-minor of the disjoint union of two graphs H_1 and H_2, and the tree-width of it is at most $2k$, and $|V(H_1)| + |V(H_2)| \leq (2k+1)|V(G)| - 6k$. Thus, we conclude the theorem. □

Proof of Theorem 3.2. We can easily obtain the proof of Theorem 3.2 from the proof of Theorem 3.1. □

4. Graphs with rank-width or linear rank-width at most 1

Distance-hereditary graphs are introduced by Bandelt and Mulder [2]. A graph G is *distance-hereditary* if for every connected induced subgraph H of G and vertices a, b in H, the distance between a and b in H is the same as in G. Oum [7] showed that distance-hereidtary graphs are exactly graphs of rank-width at most 1. Recently, Ganian [5] obtain a similar characterization of graphs of linear rank-width 1. In this section, we obtain another characterizations for these classes in terms of vertex-minor relation.

Note that every tree has rank-width at most 1 and every path has linear rank-width at most 1.

Theorem 4.1. *Let G be a graph. The following are equivalent:*

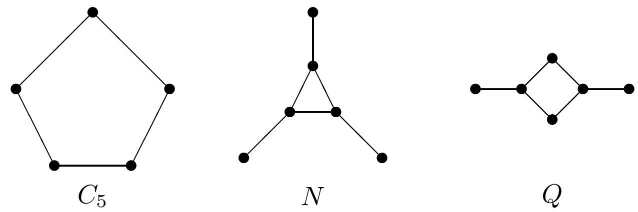

FIGURE 6. The graphs C_5, N and Q.

(1) *G has rank-width at most* 1*.*
(2) *G is distance-hereditary.*
(3) *G has no vertex-minor isomorphic to C_5.*
(4) *G is a vertex-minor of a tree.*

Proof. ((1) $\Leftrightarrow$ (2)) is proved by Oum [7], and ((2) $\Leftrightarrow$ (3)) follows from the Bouchet's theorem [3, 4]. Since every tree has rank-width at most 1, ((4) $\Rightarrow$ (1)) is trivial. We want to prove that (1) implies (4).

Let G be a graph of rank-width at most 1. We may assume that G is connected. If $|V(G)| \leq 2$, then G itself is a tree. So we may assume that $|V(G)| \geq 3$. Let (T, L) be a rank-decomposition of G of width 1. From Proposition 3.9, a rank-expansion H with the rank-decomposition (T, L) has G as a pivot-minor.

The width of each edge in T is 1. Thus for $v \in V_I(T)$, the subgraph $H[S_v]$ is a path of length 2 or a triangle because G is connected. Also for $e \in E_I(T)$, $H[\overline{e}]$ consists of an edge. Therefore H is connected and does not have cycles of length at least 4.

Let Q be a tree obtained from H by replacing each triangle abc with $K_{1,3}$ by adding a new vertex d, making d adjacent to a, b, c and deleting ab, bc, ca. Clearly H is a vertex-minor of the tree Q because we can obtain the graph H from Q by applying local complementation on those new vertices and deleting them. Therefore G is a vertex-minor of a tree, as required. $\square$

We also obtain a characterization of graphs with linear rank-width at most 1. Obstructions sets for graphs of linear rank-width 1 are C_5, N and Q [1], depicted in Figure 6.

Lemma 4.2. *Every subcubic caterpillar is a pivot-minor of a path.*

Proof. Let H be a subcubic caterpillar. By the definition of a caterpillar, there is a path P in H such that every vertex in $V(H) \setminus V(P)$ is a leaf. We choose such path $P = p_1p_2 \ldots p_m$ in H with maximum length. We construct a path Q from P by replacing each edge p_ip_{i+1} with a path $p_ia_ib_ip_{i+1}$. We can obtain a pivot-minor of Q isomorphic to P by pivoting each edge a_ib_i and deleting all a_i and deleting b_i if p_i is not adjacent to a leaf in H. $\square$

Theorem 4.3. *Let G be a graph. The following are equivalent:*

(1) *G has linear rank-width at most* 1*.*
(2) *G has no vertex-minor isomorphic to C_5, N or Q.*
(3) *G is a vertex-minor of a path.*

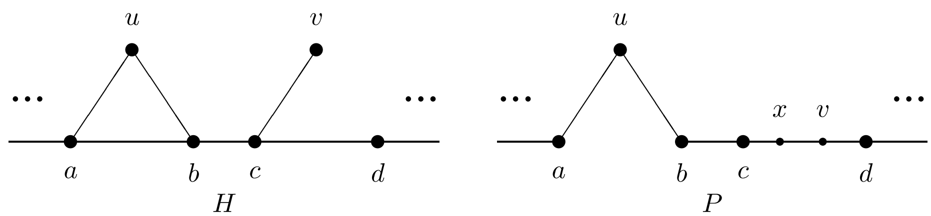

FIGURE 7. A rank-expansion H of a graph with linear rank-width 1. The graph H can be obtained from a path P by applying local complementation on u and pivoting xv and deleting x.

Proof. ((1) $\Leftrightarrow$ (2)) is proved by Adler, Farley and Proskurowski [1]. Since every path has linear rank-width at most 1, ((3) $\Rightarrow$ (1)) is trivial. Let us prove that (1) implies (3).

Let G be a graph of linear rank-width at most 1. We may assume that G is connected and $|V(G)| \geq 3$. Let H be a rank-expansion of G with a linear rank-decompostion (T, L) of width 1. Note that T is a caterpillar.

Since (T, L) is a linear rank-decomposition of width 1, for each triangle in H, one of those vertices is of degree 2 in H. Let P be a subcubic caterpillar obtained from H by replacing each triangle with a path of length 2 whose internal vertex has degree 2 in H. We can obtain H from P by applying local complementation on the inner vertex of those paths of length 2, H is a vertex-minor of P. And by Lemma 4.2, P is a pivot-minor of a path. Therefore G is a vertex-minor of a path. □

In Theorems 4.1 and 4.2, if a given graph is bipartite, we do not need to apply local complementation at some vertices. To prove it, we need the following lemma.

Lemma 4.4. *Let G be a connected bipartite graph with rank-width* 1 *and $|V(G)| \geq$ 3. Let (T, L) be a rank-decomposition of width* 1*. Then a rank-expansion of G with respect to (T, L) is a tree.*

Proof. Let $x \in V(T)$ be a leaf and H be a rank-expansion $\boldsymbol{R}(G, T, L, x, \{U_f\}_{f \in E(T)})$ of G.

Suppose that H has a triangle. Then there exists a vertex $v \in V_I(T)$ such that $H[S_v]$ is the triangle. Let e_1, e_2 and e_3 be edges incident with v and assume that e_1 is the incoming edge. Let $U_{e_1} = \{a\}$, $U_{e_2} = \{b\}$ and $U_{e_3} = \{c\}$. By the construction of a rank-expansion, $bc \in E(G)$ and $R_a^{e_1} = R_b^{e_1} = R_c^{e_1}$. Since $R_a^{e_1}$ is a non-zero vector, there is a vertex $x \in V(G)$ such that x is adjacent to all of a, b and c. Therefore xbc is a triangle in G, contradiction. □

Theorem 4.5. *Let G be a graph. Then G is bipartite and has rank-width at most* 1 *if and only if G is a pivot-minor of a tree.*

Proof. We may assume that G is connected. Since every tree has rank-width at most 1, backward direction is trivial. If G is bipartite and has rank-width at most 1, then by Lemma 4.4, we have a rank-expansion of G which is a tree. Hence, G is a pivot-minor of a tree. □

Theorem 4.6. *Let G be a graph. Then G is bipartite and has linear rank-width* 1 *if and only if G is a pivot-minor of a path.*

Proof. We may assume that G is connected. Similarly, backward direction is trivial. Suppose G is bipartite and has linear rank-width 1. Let H be a rank-expansion of G with a linear rank-decomposition (T, L) of width 1. By Lemma 4.4, the graph H is a tree, and since T is a subcubic caterpillar, H is also a subcubic caterpillar. By Lemma 4.2, H is a pivot-minor of a path, and so is G. $\square$

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